Lecture 3 : Linear Algebra (2)

$A_{1\times n}$ row vector , denoted by $\vec{v}, \vec{u}, \vec{w} \in \mathbb{R}^n$

$A_{n\times 1}$ column vector , denoted by $\vec{v}, \vec{u}, \vec{w} \in \mathbb{R}_n$

$V_{1\times n} \times U_{1\times n}$ (X)

$V_{1\times n} \times U^T_{1\times n}$ (V) $\quad t \in \mathbb{R}$ (scalar)

$\mathbb{R}_n$ (column vector), ① $0 \in \mathbb{R}_n \quad 0 = [0 \ \dots \ 0]^T$

② $\vec{v}, \vec{w}, \in \mathbb{R}_n$, $\vec{v}+\vec{w} \in \mathbb{R}_n$

③ $t \cdot \vec{v} \in \mathbb{R}_n$

} condition

$\Rightarrow$ vector space

example

$\mathbb{R}_2$ (x-y plane) $\qquad$ $\mathbb{R}_3$ (x-y-z space)

2D $\qquad$ 3D

half plane

$\mathbb{R}_2$

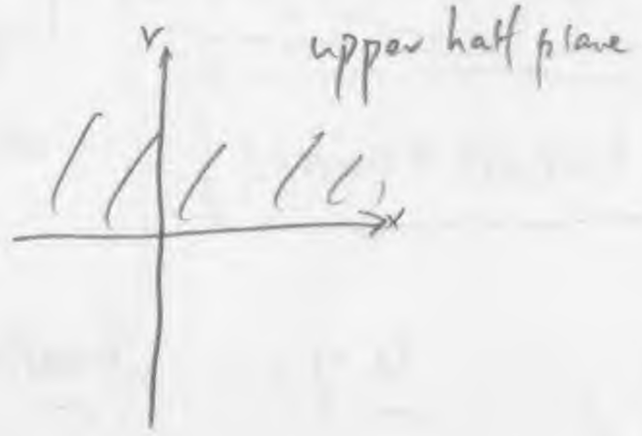
$\vec{u}+\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$t \cdot \vec{v} \in V$

$t\begin{bmatrix} x \\ y \end{bmatrix}$, $t > 0$ (V)

$t < 0 \qquad [ty] \in V$ (not a space)

so



$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $w = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$ $\in \mathbb{R}_2$

Can we $\alpha\vec{u} + \beta\vec{v} = \vec{w}$?

$2\vec{u} + 2\vec{v} = \vec{w}$ (V)

definition

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in V$

$\vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k$

v is linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$

$t_1^2 + t_2^2 + \dots + t_k^2 \neq 0$

the square sum is not equal to 0

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Can we make $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \alpha\vec{v}_1 + \beta\vec{v}_2$? Impossible

$\therefore \vec{v}_1, \vec{v}_2$ and $\vec{v}$ are linearly independent

definition

If $\vec{v} = \sum t_k \vec{v}_k$, $\sum t_k^2 \neq 0$

then $\{ v, \vec{v}_1, \dots, \vec{v}_k \}$ Linearly dependent

$\vec{v}_1, \dots, \vec{v}_k$ dependent or independent ?

$t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k = 0$

$t_1, \dots, t_k$ all zero or not

all zero $\rightarrow$ independent

not $\rightarrow$ dependent

Find solution of $t_1, \dots, t_k$

- if $t_1 = t_2 = \dots = t_k = 0$

  then $\vec{V}_1, \dots, \vec{V}_k$ are <u>linearly independent</u>

- if $t_i \neq 0$

  $\vec{V}_1, \dots, \vec{V}_k$ are <u>linearly dependent</u>

  $\vec{V_i} = t_1' V_1 + \dots + t_{i-1}' \vec{V}_{i-1} + t_{i+1}' \vec{V}_{i+1} + \dots + t_k' \vec{V}_k$

Method (for square matrix)

$$\left[\vec{V}_1, \vec{V}_2, \dots \vec{V}_k\right]_{n \times k} \begin{bmatrix} t_1 \\ \vdots \\ t_k \end{bmatrix}_{k \times 1} = 0$$

① $[t_k]$ infinity number solution

then $\vec{V}_1, \dots, \vec{V}_k$ are linear dependent

② $[t_k]$ trivial solution (homogeneous?)

then $\vec{V}_1, \dots, \vec{V}_k$ are linear independent

$\Leftrightarrow \left[\vec{V}_1, \dots, \vec{V}_k\right]$ is nonsingular (invertible)

only square matrix can be linearly dependent in this method

* non-square matrix, if it is not linearly independent in this method

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$\mathbb{R}^3$, $\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{V}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $k < 0$ $2 < 3$, we <u>cannot</u> use nonsingular to check if

* $t_1 \vec{V}_1 + t_2 \vec{V}_2 = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = 0 \quad , \text{ independent}$$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{V}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{V}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det[\,] = 0$ or not

$\rightarrow$ nonsingular $\rightarrow$ independent

if $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ singular $\rightarrow$ dependent

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$\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{V}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{V}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\vec{V}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow$ infinitely many solution (not unique solution) $\rightarrow$ always dependent

$k > n$ $4 > 3$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{V}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $\vec{V}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$ | trivial solution only $\Leftrightarrow$ A is nonsingular

Solve $A\vec{t} = 0$ (homogeneous system)

<u>Note</u>

- infinitely many solution (dependent)
- unique solution / trivial solution (independent)

$\det(A) = |A| = 0 \Leftrightarrow$ A is singular $\rightarrow$ linearly dependent

$\vec{V}_1, \vec{V}_2, \vec{V}_3$ dependent $\rightarrow$ then $\vec{V}_3 = \alpha \vec{V}_1 + \beta \vec{V}_2$

Note: <u>nonsingular</u> $\Leftrightarrow$ linearly independent

linearly independent cannot show the linear combination

$A\vec{t} = 0 \Rightarrow \vec{t} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} w$

$t_1 = 1,\ t_2 = -2,\ t_3 = 1$

$\vec{v_1} - 2\vec{v_2} + \vec{v_3} = 0$

$\Rightarrow \quad \vec{v_3} = -\vec{v_1} + 2\vec{v_2}$

Inner product → row vector × column vector

$\vec{v}, \vec{w} \in \mathbb{R}^n$ (column vector)

$\vec{v}^T \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$

$\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$

$= \langle \vec{w}, \vec{v} \rangle = \vec{w}^T \cdot \vec{v}$

- $\langle \vec{v} + \vec{u}, \vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle + \langle \vec{u}, \vec{w} \rangle$
- $t\langle \vec{v}, \vec{w} \rangle = t(v_1 w_1 + \cdots + v_n w_n)$

$= tv_1 w_1 + \cdots + tv_n w_n$

$= \langle tv, w \rangle$

$= v_1(tw_1) + \cdots + v_n(tw_n)$

$= \langle v, tw \rangle$

$\langle \vec{v}, \vec{v} \rangle = v_1^2 + v_2^2 + \cdots + v_n^2 > 0$

$\langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow v_1 = \cdots = v_n = 0$

$\Leftrightarrow \vec{v} = 0$

$t(v,w) \neq \langle tv, tw \rangle$

$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}$

$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}, \quad \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\|\vec{v}\| = \sqrt{x^2 + y^2}, \quad \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\|\vec{u}\| = 1, \quad \vec{u}$: unit vector

$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \cdot \|\vec{w}\|$

$|v_1 w_1 + \cdots + v_n w_n| \leq \sqrt{(v_1^2 + \cdots + v_n^2)(w_1^2 + \cdots + w_n^2)}$

$-1 \leq \cos\theta = \dfrac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|} \leq 1$

$\cos\theta = \left\langle \dfrac{\vec{v}}{\|\vec{v}\|}, \dfrac{\vec{w}}{\|\vec{w}\|} \right\rangle$

$\left\| \dfrac{\vec{v}}{\|\vec{v}\|} \right\| = \dfrac{\|\vec{v}\|}{\|\vec{v}\|} = 1$

eg $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\cos\theta = \dfrac{1 \times 5 + 3 \times 2}{\sqrt{1+9}\sqrt{25+4}}$

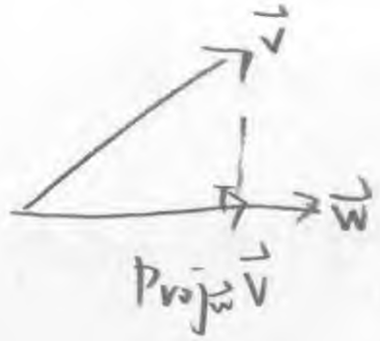
$\vec{v} \perp \vec{w} \Leftrightarrow \cos\theta = 0$

$\Leftrightarrow \langle v, w \rangle = 0$

$\langle \vec{v}, \vec{w} \rangle = 0 \quad \Leftrightarrow \quad \vec{v} \perp \vec{w}$

① $\vec{v} \perp \vec{w}$ ② $\vec{v} = 0$ ③ $\vec{w} = 0$

- $\vec{v} \neq 0$
- $\vec{w} \neq 0$



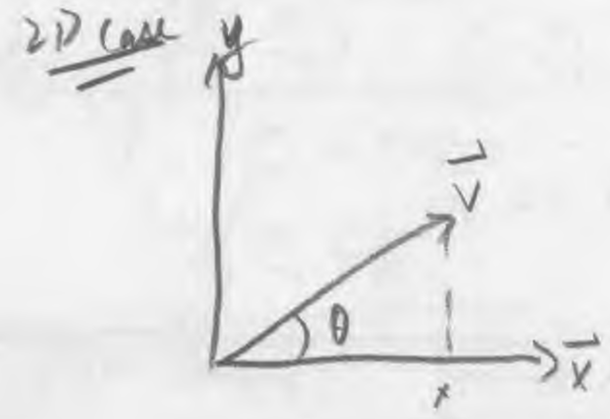
$$\text{Proj}_{\vec{w}}\vec{v} = \|\vec{v}\|\cos\theta \frac{\vec{w}}{\|\vec{w}\|} \quad \text{(unit vector (direction))}$$

$$= \|\vec{v}\| \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\|\|\vec{w}\|} \frac{\vec{w}}{\|\vec{w}\|}$$

$$= \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|} \frac{\vec{w}}{\|\vec{w}\|}$$

$$\|\text{Proj}\| = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|}$$

2D case



$x = \|\text{Proj}_{\vec{x}}\vec{v}\|$

$y = \|\vec{v}\|\sin\theta$

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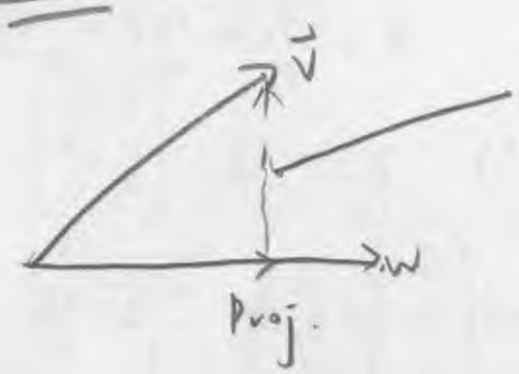
for n-D case



direction

$\vec{v} - \text{Proj}.$

$x = \|\text{Proj}_{\vec{x}}\vec{v}\|$

$y = \|\vec{v} - \text{Proj}_{\vec{x}}\vec{v}\|$

$A\vec{x} = \vec{b}$

↓

diagonal matrix → $D\vec{x} = \vec{c}$

$I\vec{x} = \vec{d} \quad \Rightarrow \vec{x} = \vec{d}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2I$$

$D_* = \lambda I$

If $A = \lambda I$ is impossible

$A\vec{v} = \lambda I\vec{v}$

$(A - \lambda I)\vec{v} = 0$

$\vec{v} \neq 0$

$(A - \lambda I) \perp \vec{v}$

$A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq 0$

$\Leftrightarrow \quad (A - \lambda I)\vec{v} = 0$, $A \neq \lambda I$

$A - \lambda I$ (singular)

$\det(A - \lambda I) = 0 \Rightarrow \lambda_i \longrightarrow (A - \lambda I)\vec{v} = 0 \rightarrow \vec{v}_i$

$\vec{v} \perp \vec{w}$

↗

$\langle v, w \rangle = 0$

$A\vec{x} = 0$

case ① / case ② ↘

$x = 0$ $A \perp \vec{x}$

$$A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0$$ ← we need it be singular

$\lambda_1 = 6$, $\lambda_2 = 1$

① $\lambda_1 = 6$ $\quad (A - \lambda_1 I)\vec{v} = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\vec{v} = 0$

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}\vec{v} = 0$$

$v_1 - 4v_2 = 0$

$v_2 = t \in \mathbb{R}$ $\Rightarrow$ $v_1 = 4t$

$$v_1 = \begin{bmatrix} 4t \\ t \end{bmatrix} , t \in \mathbb{R}$$

Select $\vec{v_1} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} / \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

② $\lambda_2 = 1 \rightarrow \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ (D: Diagonal)

$A^{100} = \underbrace{PDP^{-1}PDP^{-1}\cdots PDP^{-1}}_{100} = PD^{100}P^{-1}$

$$= P\begin{bmatrix} d_1^{100} & & \\ & d_2^{100} & \\ & & \ddots \\ & & & d_{100}^{100} \end{bmatrix}P^{-1}$$

$AP = PD = DP$ ← Diagonal matrix can move place

$A\vec{v_i} = \lambda_i \vec{v_i}$

$$A\begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$$

step $A \rightarrow |A - \lambda I| \longrightarrow \lambda^n + a\lambda^{n-1} \cdots = 0$

$\rightarrow \lambda_1, \cdots, \lambda_n \rightarrow D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$v_1 \quad \cdots \quad v_n \qquad P = \begin{bmatrix} \vec{v_1} & \cdots & \vec{v_n} \end{bmatrix}$

to diagonalise

P^{-1} , D

$$V = \begin{bmatrix} \alpha + \beta \\ 2\alpha + \beta \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Diagonalization

A square matrix A is said to be diagonalizable if there is a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$. We say that the matrix P diagonalizes A.

If $A = A^T$ Symmetric

If $Q^{-1} = Q^T$ Orthogonal

Symmetric $A = QDQ^T$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$$

$\lambda_1 = 4 \qquad \lambda_2 = 2 \qquad \|\vec{v}\| = 1$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{v}_1^* = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \qquad \vec{v}_2^* = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

normalization

$$\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\|\vec{v}_1\| = \sqrt{2} \qquad \|\vec{v}_2\| = \sqrt{2} \qquad Q = [\vec{u}_1 \ \ \vec{u}_2]$

eg $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \rightarrow \lambda_1 = 2, \ \lambda_2 = 4$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Why $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \neq Q$?

That is because we need to let $QQ^T = I$ (means the norm of the matrix should be equal to 1.)

However, $PP^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = D \neq I$

Hence, we need to normalize them to acquire $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ — unit vector

Then $Q^{-1} = Q^T$ now and we get $QQ^T = I$